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CHARACTERIZATION OF POLYNOMIALS

Abstract. In 1954 it was proved if f is infinitely differentiable in the interval I and some derivative (of order depending on x) vanishes at each x, then f is a polynomial. Later it was generalized for multi-variable case. In this paper we give an extension for distributions.

1. Introduction

In [2] it was proved that if $f : \mathbf{R} \to \mathbf{R}$, $f \in C^\infty(\mathbf{R})$, and for every $x \in \mathbf{R}$ there exists $n(x) \in \mathbf{N}$ such that $f^{(n(x))}(x) = 0$, then f is a polynomial. Later, see [1] and [3] a similar result was proved for multi-variable case.

To extend this result for distributions first we introduce some notations and recall some known results, see e.g. in [4].
Let $\Omega \subseteq \mathbf{R}^n$ be a non-empty open set. In the discussion of functions of n variables, the term multi-index denotes an ordered n-tuple

$$\alpha = (\alpha_1, \ldots, \alpha_n)$$

of nonnegative integers α_i $(i = 1, \ldots, n)$. With each multi-index α is associated the differential operator

$$D^\alpha := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

whose order is $\|\alpha\| := \alpha_1 + \ldots + \alpha_n$. If $\|\alpha\| = 0$ then $D^\alpha f := f$.

The support of a complex function f on any topological space is the closure of the set $\{x \mid f(x) \neq 0\}$.

If K is a compact set in Ω then $\mathcal{D}_K$ denotes the vector subspace of all complex-valued function $f \in C^\infty(\Omega)$ whose support lies in K, $C_0^\infty(\Omega)$ is the set of all $f \in C^\infty(\Omega)$ whose support is compact and lies in Ω. It is possible to define a topology on $C^\infty(\Omega)$ (generated by the $\|D^\alpha f\|_\infty$

norms) which makes $C^\infty(\Omega)$ into a Fréchet space (locally convex topological vector space whose topology is induced by a complete invariant metric), such that $\mathcal{D}_K$ is a closed subspace of $C^\infty(\Omega)$, τ_K denotes the Fréchet space topology of $\mathcal{D}_K$.

Choose the non-empty compact sets $K_i \subset \Omega$ $(i = 1, 2, \ldots)$ such that K_i lies in the interior of K_{i+1} and $\Omega = \cup K_i$, τ_{K_i} denotes the Fréchet space topology of $\mathcal{D}_{K_i}$. Denote τ the inductive limit topology of τ_{K_i} $(i = 1, 2, \ldots)$.

The topological vector space of test functions $\mathcal{D}(\Omega)$ is $C_0^\infty(\Omega)$ with τ. This topology is independent of the choice of K_i $(i = 1, 2, \ldots)$. A linear functional on $\mathcal{D}(\Omega)$ which is continuous with respect to τ is called a distribution in Ω. The space of all distributions in Ω is denoted by $\mathcal{D}'(\Omega)$.

In [2] the polynomiality was proved using Baire's theorem ($\mathbf{R}$ is a complete metric space). In our case $\mathcal{D}(\Omega)$ is not metrizable and the topology τ is not locally compact so we can not apply Baire's theorem to $\mathcal{D}(\Omega)$ immediately.

2. Results

In the next result we assume that Ω and K_i $(i = 1, 2, \ldots)$ are connected sets, if Ω had components then we could apply our results for each component.

Theorem 2.1. *If $u \in \mathcal{D}'(\Omega)$ and for every $\varphi \in \mathcal{D}(\Omega)$ there exists $m(\varphi) \in \mathbf{N}$ such that $(D^\alpha u)(\varphi) = 0$ for all multi-indeces α satisfying $\|\alpha\| = m(\varphi)$, then u is a polynomial (in distributional sense).*

3. Proofs

To proof the Theorem we need lemmas.

Definition 3.1. If $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbf{R}^n$ then $\mathbf{a} < \mathbf{b}$ means $a_i < b_i$, $i = 1, \ldots, n$. The set $(\mathbf{a}, \mathbf{b}) := \{\mathbf{x} \,|\, \mathbf{a} < \mathbf{x} < \mathbf{b}\}$ is the n-dimensional open interval.

Lemma 3.2. *Suppose* Γ *is an open cover of an open set* $\Omega \subseteq \mathbf{R}^n$, *and suppose that to each* $\omega \in \Gamma$ *corresponds a distribution,* $\Lambda_\omega \in \mathcal{D}'(\omega)$ *such that*

$$\Lambda_{\omega'} = \Lambda_{\omega''} \quad in \quad \omega' \cap \omega''$$

whenever $\omega' \cap \omega'' \neq \emptyset$.

Then there exists a unique $\Lambda \in \mathcal{D}'(\Omega)$ *such that*

$$\Lambda = \Lambda_\omega \quad in \quad \omega$$

for every $\omega \in \Gamma$.

Proof. See e.g. [4] 6.21 Theorem. □

Lemma 3.3. *Assume* $I \subseteq \mathbf{R}^n$ *and* $J \subseteq \mathbf{R}^m$ *are n-dimensional intervals,* $\varphi \in \mathcal{D}(I \times J)$. *Then there exists* $\varphi_k \in \mathcal{D}(I \times J)$ *such that*

$$\varphi_k = \sum_{j=1}^{k} \psi_j \chi_j,$$

where $\psi_j \in \mathcal{D}(I)$, $\chi_j \in \mathcal{D}(J)$ *and* φ_k *converges to* φ *in* $\mathcal{D}(I \times J)$.

Proof. See e.g. [5] page 111-113. □

Proof of Theorem 2.1:

In the first step we prove that for each $i = 1, 2, \ldots$ there exists a number $\gamma^{(i)} \in \mathbf{N}$ such that $D^\alpha u \equiv 0$ for all multi-indeces α satisfying $\|\alpha\| = \gamma^{(i)}$ in $\mathcal{D}_{K_i}$. Denote

$$Z^{(m)} := \{\varphi \in \mathcal{D}_{K_i} \mid (D^\alpha u)(\varphi) = 0, \|\alpha\| = m\}, \quad m \in \mathbf{N}.$$

Obviously

$$\mathcal{D}_{K_i} = \bigcup_{m \in \mathbf{N}} Z^{(m)}.$$

Here $Z^{(m)}$ is closed, because

$$Z^{(m)} = \bigcap_{\|\alpha\| = m} \{\varphi \in \mathcal{D}_{K_i} \mid (D^\alpha u)(\varphi) = 0\}$$

and $D^\alpha u$ is continuous. Since $\mathcal{D}_{K_i}$ is a complete metrizable space, Baire's theorem gives there exists $\gamma^{(i)} \in \mathbf{N}$ such that int $Z^{(\gamma^{(i)})} \neq \emptyset$ (int

is in the topology τ_i). Since $Z^{(m)}$ is a linear subspace in $\mathcal{D}_{K_i}$, we obtain $\mathcal{D}_{K_i} \equiv Z^{(\gamma^{(i)})}$.

In the second step we consider the one variable case.
If $n = 1$ then $D^{\gamma^{(i)}} u \equiv 0$ implies u is a polynomial (in distributional sense) in $\mathcal{D}_{K_i}$, see [6], Exercise 7.23, p. 99. Since $K_i \subset \operatorname{int} K_{i+1}$ and $\mathcal{D}_{K_i} \subset \mathcal{D}_{K_{i+1}}$ we obtain the theorem in this case.

In the third step we consider the multivariable case.
Since open intervals form a base for open sets in $\mathbf{R}^n$, we can write $\operatorname{int} K_{i+1} = \cup_{\omega \in \Gamma} \omega$, where Γ is an open cover of $\operatorname{int} K_{i+1}$. If we prove that u is a polynomial (in distributional sense) in each ω, then by Lemma 3.2 we obtain that u is a polynomial (in distributional sense) in $\operatorname{int} K_{i+1} \supset K_i$. Since $\mathcal{D}_{K_i} \subset \mathcal{D}_{K_{i+1}}$ we obtain the theorem.

Write $\omega := I_1 \times \cdots \times I_n$, where $I_i \subseteq \mathbf{R}$, $i = 1, \ldots, n$ are open intervals. Let $\varphi \in \mathcal{D}(\omega)$. By an obvious generalization of Lemma 3.3 there exists a sequence of $\varphi_k \in \mathcal{D}(\omega)$ such that φ_k has the form

$$\sum_{j=1}^{N} \psi_j^{(1)} \cdots \psi_j^{(n)}$$

with $N := N(k)$, where $\psi_j^{(m)} \in \mathcal{D}(I_i)$, $m = 1, \ldots, n$, $j = 1, \ldots, N$ and φ_k converges to φ in $\mathcal{D}(\omega)$. Since u is continuous, $u(\varphi_k) \to u(\varphi)$. Hence it is enough to prove, that u acts on φ_k as a polynomial (in distributional sense).

We proved that $D^{\left(\gamma^{(i)},0,\ldots,0\right)} u \equiv 0$ in $\mathcal{D}(\omega)$. It means

$$\left(D^{\left(\gamma^{(i)},0,\ldots,0\right)} u\right) \left(\psi_j^{(1)} \cdots \psi_j^{(n)}\right) \equiv 0, \quad j = 1, \ldots, N.$$

If we define $v_j \in \mathcal{D}'(I_1)$ such that

$$v_j(\psi^{(1)}) := u(\psi^{(1)} \psi_j^{(2)} \cdots \psi_j^{(n)}),$$

where $\psi^{(1)} \in \mathcal{D}(I_1)$, (we consider $\psi_j^{(2)}, \ldots, \psi_j^{(n)}$ as parameters), then we obtain $D^{\gamma^{(i)}} v_j \equiv 0$. The one variable case implies that $v_j \equiv p_j$, where p_j is a polynomial of degree at most $\gamma^{(i)} - 1$,

$$p_j(x_1) = \sum_{r=0}^{\gamma^{(i)}-1} c_r x_1^r,$$

where

$$c_r = c_r(\psi_j^{(2)}, \dots, \psi_j^{(n)}),$$

and

$$x_1 \in I_1.$$

Continuing this idea, we know that $D^{\left(\gamma^{(i)}-1,1,0,\dots,0\right)} u \equiv 0$ in $\mathcal{D}(\omega)$. From this and one variable case we obtain

$$c_{\gamma^{(i)}-1} = d_{\gamma^{(i)}-1}(\psi_j^{(3)}, \dots, \psi_j^{(n)}).$$

Similarly, we have

$$c_r(\psi_j^{(2)}, \dots, \psi_j^{(n)}) = \sum_{s=0}^{\gamma^{(i)}-1-r} d_s x_2^s,$$

where

$$d_s = d_s(\psi_j^{(3)}, \dots, \psi_j^{(n)}),$$

and

$$x_2 \in I_2.$$

Repeating this process with $D^{\left(0,\gamma^{(i)},0,\dots,0\right)} u$, $D^{\left(0,\gamma^{(i)}-1,1,0,\dots,0\right)} u$, …, $D^{\left(0,0,\gamma^{(i)},0,\dots,0\right)} u$, …, $D^{\left(0,\dots,0,\gamma^{(i)}\right)} u$ we obtain that u acts on φ_k as a polynomial (in distributional sense) indeed, and the proof of the theorem has completed. ■

References

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